VIII. Conclusion

Methods for systematically studying optimum two-impulse orbital transfer between any pair of elliptical orbits have been developed and used to investigate the properties of an "impulse function space." The mechanisms that govern the structure of this function space have been determined and may be employed now to predict the optimum transfers resulting from any pair of elliptical orbits.

A contouring technique was used to present large amounts of optimum impulse information in a concise, easily understood form. The results obtained from function contouring have been verified through the use of steepest descent optimization procedures.[2,8] Initial conditions taken from the contour maps always allowed the numerical search program to converge to the proper local optimum within a few seconds of IBM 7090 time. Optimum impulses obtained from this exact numerical optimization were only slightly better than those obtained from contouring. In all instances where exact numerical solutions were required, the insight gained through contouring methods proved to be an invaluable aid to subsequent optimization by conventional techniques.

Although the methods presented here are oriented principally toward the area of space mission design, they also provide numerous clues that point the way to analytical solution of numerous subproblems.

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Effect of an Oblate Rotating Atmosphere on the Eccentricity, Semimajor Axis, and Period of a Close Earth Satellite

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Expressions are derived for the secular effects of a rotating atmosphere on the eccentricity of a close earth satellite for $0 \leq e \leq 0.01$. Two cases were considered: 1) the atmosphere is spherical and its density varies exponentially with altitude; and 2) the atmosphere is oblate, has the same flattening as the earth, and varies exponentially with altitude. In both cases, the atmosphere was assumed to rotate with the same angular velocity as the earth. In addition, expressions are derived for the secular effects of the oblate rotating atmosphere on the semimajor axis and period for $0 \leq e \leq 0.01$. Variations in air density, such as diurnal or seasonal variations, were neglected. It was found that both the rotation of the atmosphere and its oblateness significantly affect the eccentricity, semimajor axis, and period over one revolution. The expressions derived are accurate for the cases being considered and can be used for accurate atmospheric density interpretations based on known satellite motions or for accurate predictions of satellite motion and position. The modified Bessel functions of the first kind are used extensively in this report. Hence, for purposes of completeness, they are tabulated up to the eighth order for an argument of 0 through 2.

Nomenclature

a = semimajor axis
A = satellite cross-sectional area
A_0,A_1,A_2,A_3,A_4,A_5 = quantities which are defined in Eqs. (75–80)
B = ballistic coefficient = $C_DA/2m$
B_0,B_1,B_2,B_3,B_4,B_5 = quantities defined in Eqs. (94–99)
c = $K a e$
C_D = aerodynamic drag coefficient
d = $(\Omega_e/n)(1 - e^2)^{1/2} \cos i$
D_S = radial component of drag force per unit mass
D_T = component of drag force per unit mass perpendicular to the radius vector
D_W = component of drag force per unit mass normal to the orbital plane
$\mathbf{D}$ = $-B\rho v^2\hat{v}$ = aerodynamic drag force per unit mass due to air resistance
e = orbital eccentricity
E = eccentric anomaly
f = earth's flattening = 1/298.3
h = altitude above the surface of a spheroidal earth; also the aereal constant
h_π = altitude at perigee

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i	=	orbital inclination measured from due east on the equator at the ascending node to the orbital plane, so that $0° \leq i \leq 180°$
I_v	=	modified Bessel functions of the first kind of order v
j	=	$[(1 + d)/(1 - d)]$
K	=	$-(d/dz) \log_\epsilon \rho$
m	=	mass of the satellite
n	=	$2\pi/\tau$ = satellite's mean motion
q	=	$KR_e f \sin^2 i$
q_0, q_1, q_2, q_3, q_4	=	quantities defined by Eqs. (69–73)
r	=	orbital radius or radius vector from the center of the earth to the satellite
R_e	=	mean equatorial radius = 3443.93 naut miles = 2.09257×10^7 ft
t	=	time and is reckoned from perigee
v	=	satellite's air-velocity
z	=	satellite's height above perigee
β	=	central angle in orbital plane from ascending node to the satellite, measured in direction of satellite motion
ϵ	=	base of the natural logarithms = 2.71828
θ	=	true anomaly, the orbital angle from perigee to satellite
μ	=	gravitational constant = 1.407645×10^{16} ft^3/sec^2
π	=	3.14159
ρ	=	atmospheric density
ρ_π	=	atmospheric density at perigee
τ	=	anomalistic period, i.e., the time in orbit from perigee to perigee
ω	=	central angle to perigee
Ω_e	=	earth's angular velocity about its axis = 4.178074×10^{-3} deg/sec

Introduction

A CLOSE earth satellite experiences forces due to both an oblate, rotating atmosphere and the gravitational potential of an oblate earth. It is fairly well known how the gravitational perturbations, due to the earth's oblateness, secularly affect the various quantities, such as the orbital period τ, position of the nodes Ω, and position of the line of apsides. In addition, the orbital elements not only vary due to neutral atmospheric drag effects, but the expressions for their secular variations contain, either explicitly or implicitly, the orbital eccentricity e and semimajor axis a.

Hence, an accurate knowledge of the secular variations $\dot{e}$ and $\dot{a}$ is needed for an accurate prediction of both the satellite's position with time and the frequency with which it can land or be called down at a given landing site.

Several authors[1–6] have studied the effect of atmospheric drag on the eccentricity of a close earth satellite. However, with one exception,[4] it is believed that the solutions obtained by these authors for the secular decay rate of e and a are less accurate than would be desirable for the purpose of accurate predictions of the satellite's position and orbital departure frequency intervals (call-down frequency). This is because of the assumptions and/or approximations used in their treatments.

For example, Garfinkel[1] assumes 1) that the resisting drag force is tangent to the path of motion and 2) that the oblateness of the earth can be neglected. He considers the case where the eccentricity of the orbit is small. Kork[2] treats the problem 1) for a spherical, nonrotating atmosphere and 2) for the case in which the eccentricities of the orbit are large. Roberson,[3] in his treatment, assumed that no forces act on the satellite, except those of air drag, and he assumed a spherical earth. He used the variation of parameters and the Krylov-Bogoliuboff approximation method to find simple approximate expressions for the decay of eccentricity with radius, the decay of radius with true anomaly, and the growth of true anomaly with time.

Perkins,[5] in arriving at "An Analytical Solution for Flight Time of Satellites in Eccentric and Circular Orbits," does not derive an expression for $\dot{e}$. However, he derives expressions for the time rate of fall of the perigee and apogee, from which an expression for the time rate of change of eccentricity e can be obtained, using an expression[2] that gives $\dot{e}$ as a function of the time rate of fall of the perigee and apogee. Although, as Perkins points out, his analytical solutions are accurate enough for most engineering purposes, more accurate formulas would be desirable. In addition, the results obtained by Perkins involve two infinite series that must be computed in the computations of $\dot{e}$, hence, the solution obtained in this report appears to be simpler and is believed to be more accurate.

King-Hele[6] gives a very excellent derivation for the effects of drag by an oblate atmosphere on elliptic orbits for two cases: $e \gtrsim 0.025$ and $e \lesssim 0.025$. In essence, he derives expressions for e as a function of 1) the initial value of e_1, 2) time in orbit after the time at $e = e_1$, and 3) the approximate satellite lifetime after $e = e_1$. He neglects the rotation of the atmosphere, but how much the rotation of the atmosphere contributes is not known accurately. On the basis of the work done in this report, it appears likely that the atmosphere's rotation is not negligible.

Sterne[4] also treats this problem of the secular variations of the orbital parameters of a close earth satellite. His treatment appears to be more rigorous and accurate than the other analyses reported in the literature, because Sterne considers the effects of an oblate, rotating atmosphere. However, the expression he obtained for the secular time rate of decay of orbital eccentricity is for eccentricities larger than about 0.01.

In addition, it appears likely that his results are not accurate for satellite altitudes of less than 200 naut miles, because, in his series expansion of the exponential form of the atmosphere, he retains only terms through q^2, assuming that for the earth q is of the order of 0.2 or less. This is true only for satellite altitudes greater than about 200 naut miles.

In this work the derivations are extended to include terms through q^4, so that the results of this work are quite accurate and/or applicable for satellite altitudes of 86 naut miles where q can be as large as 0.9. The purpose and scope of this report is to supplement Sterne's results by deriving an accurate expression for the secular time rate of decay of e and a due to the drag effects of an oblate rotating atmosphere for the case of small eccentricites, i.e., $0 \leq e \leq 0.01$ and for altitudes as low as about 86 naut miles, by using Sterne's approach. It should be pointed out, however, that any density variations, such as the daytime-nighttime variations, are neglected. King-Hele[6] discusses these effects to some extent.

It should be pointed out that Brouwer and Hori[11] have evaluated the effect of atmospheric drag on the motion of an artificial satellite in the gravitational field of an oblate earth. They included the drag forces in the equations of motion and then proceeded to solve these equations. This is the more rigorous method of determining the drag effects. However, they assumed a spherical, nonrotating atmosphere and the solution is quite lengthy.

In this work, it is assumed tacitly that the effects of drag can be linearly superimposed upon the effects of the potential perturbations to first order. However, the results obtained are relatively simple and illuminating, in that they show clearly the effect of both the oblateness and the rotation of the atmosphere.

Theory

In the following theoretical analyses, the problem of a spherical, rotating atmosphere will be treated first. Then, the problem of an oblate, rotating atmosphere will be treated. The purpose of making the former analysis is to compare the results for the two cases and to determine the significance of the atmosphere's oblateness as it affects the orbital eccentricity.

Effect of Drag by a Circular Rotating Atmosphere on Eccentricity

For this case, the following assumptions are made. The atmosphere is spherical. The atmosphere rotates with the earth about its axis and at the same angular velocity as the earth. The orbital eccentricity is small, i.e., $0 \leq e \leq 0.01$.

Satellite Airspeed

The satellite airspeed v must be evaluated first, because the drag force per unit mass D depends on v, as follows:

$$\mathbf{D} = -B\rho v\mathbf{v} \tag{1}$$

The ballistic coefficient B is given by

$$B = C_D A/2m \tag{2}$$

If the atmospheric rotation were neglected, then the satellite air velocity would be just the normal satellite velocity v_0 with respect to inertial space. This is given by

$$v_0{}^2 = \frac{\mu}{a}\frac{1 + e\cos E}{1 - e\cos E} \text{ (for no atmospheric rotation)}$$

$$= \dot{r}^2 + r^2\dot{\theta}^2 \tag{3}$$

However, because of the atmosphere's rotation, the satellite air velocity is given by

$$\mathbf{v} = \mathbf{v}_0 - \mathbf{v}_{\text{atm}} \tag{4}$$

where

$$\mathbf{v}_{\text{atm}} = \mathbf{\Omega}_e \times \mathbf{r} = \text{velocity of the atmosphere} \tag{5}$$

The following unit vectors will be defined now:

S = unit vector along the satellite radius vector $\mathbf{r}$

T = unit vector perpendicular to $\mathbf{r}$ and in the direction of the satellite motion

W = unit vector normal to the orbital plane in such a sense that the satellite appears to move clockwise when viewed along W, i.e., when the fingers of the right hand point in the direction of satellite motion, the thumb points in the direction of W

Then the atmosphere's velocity becomes

$$\mathbf{v}_{\text{atm}} = S(\Omega_T r_W - \Omega_W r_T) + T(\Omega_W r_S - \Omega_S r_W) + W(\Omega_S r_T - \Omega_T r_S) \tag{6}$$

where the S, T, and W subscripts are used to denote the S, T, and W components of $\mathbf{\Omega}_e$ and $\mathbf{r}$, so that

$$r_S = r \tag{7}$$

$$r_T = r_W = 0 \tag{8}$$

Now, $\mathbf{\Omega}_e$ has a component $\Omega_e \cos i$ in the direction of W and a component $\Omega_e \sin i$, which is in the orbital plane and perpendicular to the line of nodes.

Hence, the S, T, and W components of Ω_e are

$$\Omega_S = \Omega_e \sin i \sin\beta \tag{9}$$

$$\Omega_T = \Omega_e \sin i \cos\beta \tag{10}$$

$$\Omega_W = \Omega_e \cos i \tag{11}$$

so that

$$\mathbf{v}_{\text{atm}} = T\Omega_W r - W\Omega_T r \tag{12}$$

Here, β is the central angle from the line of nodes to the satellite, measured positive in the direction of satellite motion, whereas i is the inclination angle between the equatorial and orbital planes, measured at the ascending node, from due east on the equator to the orbital plane. Thus, $0° \leq i \leq 180°$.

The *total* satellite air velocity $\mathbf{v}$ is given by

$$\mathbf{v} = Sv_S + Tv_T + Wv_W \tag{13}$$

and

$$v^2 = v_S{}^2 + v_T{}^2 + v_W{}^2 \tag{14}$$

where

$$v_S = \dot{r} \tag{15}$$

$$v_T = r\dot{\theta} - \Omega_e r \cos i \tag{16}$$

$$v_W = \Omega_e r \sin i \cos\beta \tag{17}$$

In the forementioned analysis, the atmospheric velocity increases the satellite airspeed when it is against the satellite motion, and vice versa [see Eq. (4)]. Hence,

$$v^2 = \dot{r}^2 + r^2\dot{\theta}^2 - 2r^2\dot{\theta}\Omega_e \cos i + \Omega_e{}^2 r^2(\cos^2 i + \sin^2 i \cos^2\beta) \tag{18}$$

As shown earlier, the first two terms are just the square of the normal satellite velocity, as given by Eq. (3). Moreover, as pointed out by Sterne[4]

$$r^2\dot{\theta} = [\mu a(1 - e^2)]^{1/2} = \text{areal constant} \tag{19}$$

$$n^2 a^3 = \mu, \quad n = \text{mean motion} \tag{20}$$

$$nt = E - e \sin E, \quad t = \text{time and is reckoned from perigee} \tag{21}$$

$$r = a(1 - e\cos E) \tag{22}$$

Making use of Eqs. (19) and (20) in Eq. (18), solving for v and dropping second-order corrections to v, Sterne has shown that v is given to an accuracy of one part in five hundred

$$v = v_0\left(1 - d\,\frac{1 - e\cos E}{1 + e\cos E}\right) \tag{23}$$

where

$$d = (\Omega_e/n)\,(1 - e^2)^{1/2} \cos i \tag{24}$$

and

$$v_0 = \left[\frac{\mu}{a}\frac{1 + e\cos E}{1 - e\cos E}\right]^{1/2} \tag{25}$$

Drag Forces

The vector drag force per unit mass, as given by Eq. (1), is

$$\mathbf{D} = -B\rho v\mathbf{v} = SD_S + TD_T + WD_W = -B\rho v(Sv_S + Tv_T + Wv_W) \tag{26}$$

Hence, it follows that

$$D_S = -B\rho v v_S \tag{27}$$

$$D_T = -B\rho v v_T \tag{28}$$

$$D_W = -B\rho v v_W \tag{29}$$

where v_S, v_T, and v_W are given by Eqs. (15–17), respectively. It will be noted that here the ballistic coefficient is assumed to be constant, which is to say that the satellite frontal and side cross-sectional areas are the same. This would be strictly true only in the case of a spherical satellite.

As will be seen later, it would be advantageous to write the drag force components D_S and D_T as functions of E for purposes of integration. In addition, Plummer[7] has shown that the time derivative of eccentricity is independent of the D_W drag force component. Henceforth, the D_W component of drag force will be disregarded.

The next step, then, is to write the D_S and D_T components of drag force per unit mass as functions of E. To do this, the author will proceed as follows. From Eqs. (21) and (22), respectively, it can be seen that

$$n\dot{t} = n = (1 - e\cos E)\dot{E} \tag{30}$$

$$\dot{r} = ae \sin E\dot{E} \tag{31}$$

Hence,

$$v_S = \dot{r} = ae \sin E\dot{E} \tag{32}$$

Also, substituting for μ from Eq. (20) into Eq. (19) yields

$$r^2\dot{\theta} = na^2(1 - e^2)^{1/2} \tag{33}$$

Dividing through by r, substituting for r from Eq. (22), and substituting for n from Eq. (30), then,

$$r\dot{\theta} = \dot{E}a(1 - e^2)^{1/2} \tag{34}$$

In addition, after some mathematical manipulation, it can be shown that

$$\Omega_e r \cos i = d\,\frac{a(1 - e\cos E)^2}{(1 - e^2)^{1/2}}\,\dot{E} \tag{35}$$

The sum of Eqs. (34) and (35) is just v_T. Then, combining these equations gives

$$v_T = a(1 - e^2)^{1/2}\left[1 - d\,\frac{(1 - e\cos E)^2}{1 - e^2}\right]\dot{E} \tag{36}$$

and Eqs. (27) and (28) become

$$D_S = -B\rho vae \sin E\dot{E} \tag{37}$$

$$D_T = -B\rho va(1 - e^2)^{1/2}\left[1 - d\,\frac{(1 - e\cos E)^2}{1 - e^2}\right]\dot{E} \tag{38}$$

Secular Variation in Eccentricity

Plummer (Ref. 7, pp. 147–151) has shown that the time derivative of eccentricity, with a slight change of notation, is given by

$$de/dt = (a/u)^{1/2}\cos\phi\,[D_S \sin\theta + D_T(\cos\theta + \cos E)] \tag{39}$$

where

$$\sin\phi \equiv e \tag{40}$$

as defined by Plummer, hence,

$$\cos\phi = (1 - e^2)^{1/2} \tag{41}$$

$$\cos\theta = (\cos E - e)/(1 - e\cos E) \tag{42}$$

$$\sin\theta = (1 - e^2)^{1/2}\sin E/(1 - e\cos E) \tag{43}$$

Substituting for D_S, D_T, $\cos\phi$, $\cos\theta$, and $\sin\theta$, from Eqs. (37, 38, and 41–43) into Eq. (39) gives

$$\frac{de}{dt} = -(1 - e^2)\left(\frac{a}{\mu}\right)^{1/2} vB\rho a\dot{E}\left\{\frac{e\sin^2 E}{1 - e\cos E} + \left[1 - d\,\frac{(1 - e\cos E)^2}{1 - e^2}\right]\left[\frac{\cos E - e}{1 - e\cos E} + \cos E\right]\right\} \tag{44}$$

Now, the terms within the brackets, when expanded, and using $\sin^2 E = 1 - \cos^2 E$ give

$$\{\;\} = \left\{\frac{e(1 - \cos^2 E)}{1 - e\cos E} + \frac{\cos E - e}{1 - e\cos E} + \cos E - \frac{d}{(1 - e^2)}(1 - e\cos E)(\cos E - e) - \frac{d}{(1 - e^2)}(1 - e\cos E)^2\cos E\right\} \tag{45}$$

The sum of the first two terms of Eq. (45) are just $\cos E$. The sum of the first three terms is then $2\cos E$. Factoring $[d/(1 - e^2)](1 - e\cos E)$ in the remaining terms of Eq. (45), it becomes

$$\{\;\} = 2\cos E - \frac{2d}{2(1 - e^2)}(1 - e\cos E)(2\cos E - e - e\cos^2 E) \tag{46}$$

Substituting for v from Eq. (23) and for v_0 from Eq. (3), then Eq. (44) becomes

$$\frac{de}{dt} = -2(1 - e^2)B\rho a\left(\frac{1 + e\cos E}{1 - e\cos E}\right)^{1/2} \times \left(1 - d\,\frac{1 - e\cos E}{1 - e\cos E}\right)\left\{\cos E - \frac{d}{2(1 - e^2)}(1 - e\cos E) \times (2\cos E - e - e\cos^2 E\right\}\frac{dE}{dt} \tag{47}$$

To get the change in e over one revolution, i.e., Δe/rev, integrate Eq. (47) with respect to time over one period. As pointed out by Sterne, the integral always can be evaluated numerically for any assumed atmospheric model. However, a fairly good analytical approximation can be obtained by assuming an exponential atmosphere about a spherical earth, which obeys

$$\rho = \rho_\pi \epsilon^{-Kz} \tag{48}$$

ϵ = base of the natural logarithms
ρ_π = atmospheric density at perigee
z = height of satellite above the perigee altitude
$= r - r_p$
$= a(1 - e\cos E) - a(1 - e)$

$$= ae\,(1 - \cos E) \tag{49}$$

$K = -(d/dz)\log_\epsilon\rho$, i.e., negative of slope of $\log_\epsilon\rho$ vs z curve, sometimes called the lapse rate

As pointed out, e is assumed to be small, $0 \leq e \leq 0.01$. Hence, the radius at apogee is 2% greater than the radius at perigee, which is at most ~80 naut miles higher than the radius at perigee for a low-altitude satellite in the earth's atmosphere, and the maximum altitude range Z, which a satellite experiences oribiting an oblate earth, is about 90 naut miles for a polar orbit. The negative logarithmic slope of the density, K, does not change very rapidly with altitude, except in the altitude range below 105 naut miles as can be seen from Fig. 1.[13, 14] It can be treated as a constant over one revolution. This helps to simplify the integration of Eq. (37).

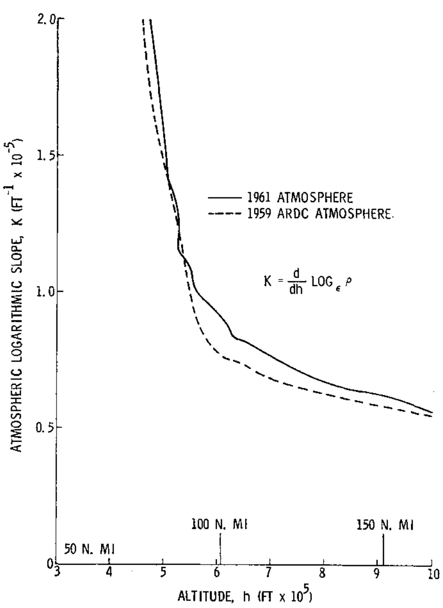

Fig. 1 Logarithmic slope, K, vs altitude.

The value of K may be the average value or a best-fit value over the altitude range encountered during that revolution or, possibly, the value at perigee. It is important that the density thus computed be correct near perigee, since the drag effects are concentrated in that region of the orbit.

Letting

$$c = Kae \tag{50}$$

and substituting for ρ from Eq. (48), then

$$\frac{\Delta e}{\text{rev}} = -2(1-e^2)\,Ba\rho_\pi\,\epsilon^{-c}\int_0^{2\pi}\epsilon^{c\,\cos E}\left[\frac{1+e\cos E}{1-e\cos E}\right]^{1/2}\times \left[1-d\frac{1-e\cos E}{1+e\cos E}\right]\left\{\cos E-\frac{d}{2(1-e^2)}\times (1-e\cos E)\,(2\cos E-e-e\cos^2 E)\right\}dE \tag{51}$$

Simplify the integrand of Eq. (51) further by using the fact that

$$d \simeq \tfrac{1}{15}\cos i \qquad 0 \leq e \leq 0.01 \tag{52}$$

Then, by binomial expansion, dropping term $0(e^3)$, and integrating,

$$\Delta e/\text{rev} = -(1-e^2)4\pi B\rho_\pi a\epsilon^{-c}[(2d+1)eI_0(c) + \{(1-d)^2-[(3d+2)/2Ka]\}\,I_1(c)] \tag{53}$$

for $e \leq 0.01$ and a spherical rotating atmosphere, where $I_v(c)$ are the modified Bessel functions of the first kind of order v.

One of the criteria for the correctness of Eq. (53) is that Δe/rev approaches zero, as e approaches 0, and in fact, Δe/rev does converge to zero as e becomes zero, because at $e = 0$, $C = 0$, and $I_0(0) = 1$, $I_1(0) = 0$.

Effect of Drag by an Oblate Rotating Atmosphere on Eccentricity

The satellite air velocity and drag forces are the same as given in the subsections on Airspeed and Drag Forces, respectively. Hence, the time rate of change of eccentricity is given by Eq. (47), as before, but now the density for an oblate atmosphere is given by

$$\rho = \rho_\pi\,\epsilon^{-K(h-h_\pi)} \tag{54}$$

where the h and h_π are the altitudes above the surface of an oblate, spheroidal earth for the satellite and perigee, respectively. Now, the radius is given very accurately[10] by

$$R = R_e(1-f\sin^2 i\sin^2\beta) \tag{55}$$

where

R_e = 3443.93 naut miles = equatorial radius of the earth
f = 1/298.3 = earth's flattening

Hence,

$$h = r - R = a(1-e\cos E) - R_e(1-f\sin^2 i\sin^2\beta) \tag{56}$$

$$h_\pi = a(1-e) - R_e\,(1-f\sin^2 i\sin^2\omega) \tag{57}$$

$$h - h_\pi = ae(1-\cos E) + R_e f\sin^2 i\,(\sin^2\beta-\sin^2\omega) \tag{58}$$

$$\rho = \rho_\pi\exp[-c\,(1-\cos E) - q\,(\sin^2\beta-\sin^2\omega)] \tag{59}$$

where

$$q = KR_e f\sin^2 i \tag{60}$$

with

$$q_{\max} \cong 0.5\sin^2 i \tag{61}$$

for low-altitude (~100 naut miles) earth satellites, and ω is the central angle to perigee. It can be seen readily that this expression for ρ would render the integration of Eq. (47) in closed form somewhat difficult.

However, Sterne gives a method for approaching this problem which proves to be quite helpful, i.e., Eq. (59), the expression for ρ, can be written as

$$\rho = \rho_\pi\epsilon^{-c(1-\cos E)}\sum_{S=0}^{\infty}(-1)^S q^S\frac{(\sin^2\beta-\sin^2\omega)}{s!} \tag{62}$$

The series converges for $q < 1$. This is true for any earth satellite, so long as $KR_e f < 1$. Using the 1959 Air Research and Development Command atmosphere, this is true at perigee altitudes in excess of 80 naut miles at $i = 90°$ and at lower altitudes for other inclination angles.

Since

$$\beta = \omega + \theta \tag{63}$$

it follows that

$$\begin{aligned}\sin^2\beta &= \sin^2(\omega+\theta)\\ &= \tfrac{1}{2} - \tfrac{1}{2}\cos(2\omega+2\theta)\\ &= \tfrac{1}{2} - \tfrac{1}{2}\cos 2\omega\cos 2\theta + \tfrac{1}{2}\sin 2\omega\sin 2\theta\end{aligned} \tag{64}$$

$$\sin^2\beta - \sin^2\omega = \cos 2\omega\sin^2\theta + \tfrac{1}{2}\sin 2\omega\sin 2\theta \tag{65}$$

Hence, in the series of Eq. (62), the terms that are odd functions of θ are also odd functions of E. They may be ignored, because they do not contribute to the complete integral taken over one revolution, i.e., over one complete cycle in E.

Sterne[4] retains only terms through q^2 in the series expansion of ρ. In the case here, the series expansion of the atmospheric density is an alternating series. Hence, according to the remainder theorem,[9] "the absolute value of the remainder after n terms of such a series does not exceed the absolute value of the $(n+1)$st term."

Hence, the relative error introduced in the series expansion of the atmospheric density by retaining only terms through q^n is

$$\text{relative error in }\rho \lessapprox \frac{q^{n+1}/(n+1)!}{e^{-q}} \tag{66}$$

Thus, by retaining terms through q^2, the relative error in ρ is 3.4% at altitudes of 100 naut miles where $q \sim 0.5$ and only 0.16% at altitudes of 200 naut miles where $q \sim 0.2$.

It might then be expected that, since ρ appears in the integrand of the expression for Δe/rev, the resulting expression for Δe/rev after integration would be in error by the same amount. Hence, the relative error in the eccentricity that would be predicted after one revolution (say the nth rev) as a result of the forementioned error is given by

$$\text{relative error in } e_{n+1} \leq \frac{\text{error in }\Delta e/\text{rev}}{e_n + \Delta e/\text{rev}} \tag{67}$$

At 100 naut miles this error is

$$\sim [(\Delta e/\text{rev})/(e+\Delta e/\text{rev})]\times(3.4\%)$$

and is probably very small because of the factor $[(\Delta e/\text{rev})/(e+\Delta e/\text{rev})]$.

When greater accuracy is desired in the expression for the Δe/rev, then terms through q^4 must be retained, in which case the expression for atmospheric density ρ (oblate atmosphere) becomes

$$\rho = \rho_\pi\epsilon^{-c}\,\epsilon^{c\,\cos E}\sum_{v=0}^{4} q_v\frac{\sin^{2v}E}{(1-e\cos E)^{2v}} \tag{68}$$

where only the even functions of E are retained, because the odd functions do not contribute to the Δe/rev integral, and where

$$q_0 = 1 \tag{69}$$

$$q_1 = (1-e^2)^2[-q\cos 2\omega + (q^2/2)\sin^2 2\omega] \tag{70}$$

$$q_2 = (1-e^2)\,[(q^2/2)\cos 4\omega - (q^3/2)\cos 2\omega\sin^2 2\omega + (q^4/24)\sin^4 2\omega] \tag{71}$$

$$q_3 = (1 - e^2)^3[-(q^3/6)\cos^3 2\omega + (q^3/2)\cos 2\omega \sin^2 2\omega + (q^4/24)\cos^2 2\omega \sin^2 2\omega - (q^4/12)\sin^4 2\omega] \quad (72)$$

$$q_4 = (1 - e^2)^4[(q^4/24)\cos^4 2\omega - (q^4/16)\sin^2 4\omega + (q^4/24)\sin^4 2\omega] \quad (73)$$

Then the expression for Δe/rev, when terms $O(10^{-6})$ are retained, for the case $e \leq 0.01$, becomes

$$\frac{\Delta e}{\text{rev}} = -(1 - e^2)4\pi B\rho_\pi a\epsilon^{-c} \sum_{v=0}^{5} A_v I_v(c) \quad (74)$$

where

$$A_0 = e(2d + 1) \quad (75)$$

$$A_1 = (1 - d)^2 - [(3d + 2)/2Ka] + (q_1/Ka)(3 - 2d) \quad (76)$$

$$A_2 = (q_1/c)[(1 - d)^2 - (3/2Ka)(6 - 5d)] \quad (77)$$

$$A_3 = (q_1/c^2)(\tfrac{33}{2}e^2) + (3q_2/c^2)[(1 - d)^2 + (1/2Ka)(10 + 17d)] + (q_3/c^2)(15/Ka)(7 - 10d + 6d^2) \quad (78)$$

$$A_4 = (3q_2/c^2)[(97/2Ka) + e(5 - 4d)] + (15q_3/c^3)[(1 - d)^2 - (7/Ka)(7 - \tfrac{21}{2}d + 6d^2) + e^2(\tfrac{55}{2} - 30d + 21d^2)] + (q_4/c^3)(105/Ka)(9 - 14d + 8d^2) \quad (79)$$

$$A_5 = -(q_3/c^4)(105e^2)[\tfrac{55}{2} - 33d + 21d^2] + (q_4/c^4)(105)[(1 - d)^2 - (9/Ka)(9 - \tfrac{29}{2}d + 8d^2) + e^2(\tfrac{89}{2} - 56d)] \quad (80)$$

To show that the (Δe/rev) converges to zero at $e = 0$, i.e., at $c = Kae = 0$, the expression for (Δe/rev) can be rewritten in the form

$$\frac{\Delta e}{\text{rev}} = -(1 - e)^2\, 4\pi B\rho_\pi a\epsilon^{-c} \sum_{v=0}^{5} A_v' \frac{I_v(c)}{c^{(v-1)}} \quad (81)$$

where $[1/(c^{v-1})$ has been factored out of the A's, so that

$$A_v = A_v'\,[1/c^{(v-1)}] \quad (82)$$

At $c = 0$, all $[I_v(c)/c^{v-1}] = 0$, while the A_v' are either zero or finite, and hence (Δe/rev) $= 0$.

In a circular orbit ($e = 0$) the location of perigee ω becomes undefined, as well as the true anomaly θ and eccentric anomaly E, which are measured from perigee. Thus, the differential equation, Eq. (39), is not unique for $e = 0$. However, the integral of this equation, (Δe/rev) $\rightarrow 0$ as $e \rightarrow 0$ for both a spherical and an oblate rotating atmosphere. Hence, an initially circular orbit will decay as a circular orbit.

For completeness, the modified Bessel function of the first kind, $I_v(c)$ are tabulated in Table 1, up through the eighth order for argument from 0 to 2.

Effect of Drag by an Oblate, Rotating Atmosphere on Semimajor Axis and Orbital Period

It has been shown by Plummer[7] that the time rate of variation of the semimajor axis is given by

$$da/dt \equiv 2(a^3/\mu)^{1/2}\,[D_S \tan\phi \sin\theta + D_T \sec\phi(1 - e\cos\theta)] \quad (83)$$

where

$$\sin\phi \equiv e \quad (84)$$

$$\sin\theta = (1 - e^2)^{1/2}\,[\sin E/(1 - e\cos E)] \quad (85)$$

$$\cos\theta = (\cos E - e)/(1 - e\cos E) \quad (86)$$

Then, substituting into Eq. (83) from Eqs. (37, 38 and 84–86),

$$\frac{da}{dt} = -2B\,\rho a^2(1 - d)^2 \frac{(1 + je\cos E)^2}{(1 - e^2\cos^2 E)^{1/2}}\dot{E} \quad (87)$$

where

$$j = (1 + d)/(1 - d) \quad (88)$$

Since

$$\tau = [2\pi/(\mu)^{1/2}]a^{3/2} = \text{unperturbed orbital period} \quad (89)$$

then

$$\dot{\tau} = \tfrac{3}{2}(\tau/a)\dot{a} \quad (90)$$

Substituting for $\dot{a}$ from Eq. (87) and integrating over one orbital period to get the $\Delta\tau$/rev, one obtains

$$\frac{\Delta\tau}{\text{rev}} = -3\tau\beta a(1 - d)^2 \int_0^{2\pi} \rho \frac{(1 + je\cos E)^2}{(1 - e^2\cos^2 E)^{1/2}} dE \quad (91)$$

Since only the $e \leq 0.01$ are being considered, the cases where $0 < e \leq 0.01$ and $e = 0$ will be treated independently, in that order.

***Case I:* $0 < e \leq 0.01$**

Using the expression for the density of an osculating exponential atmosphere, as given by Eq. (68), expanding the integrand of Eq. (91) by binomial expansion to facilitate the integration, and dropping terms $<0(e^3)$, then the $\Delta\tau$/rev and Δa/rev after integration become

$$\Delta\tau/\text{rev} = \tfrac{3}{2}(\tau/a)(\Delta a/\text{rev}) \quad (92)$$

$$\frac{\Delta a}{\text{rev}} = -4\pi Ba^2(1 - d)^2\rho_\pi\epsilon^{-c} \sum_{v=0}^{5} B_v I_v(c) \quad (93)$$

where

$$B_0 = 1 + e^2(j^2 + \tfrac{1}{2}) \quad (94)$$

$$B_1 = 2je - (e^2/c)(j^2 + \tfrac{1}{2}) + (q_1/c) \times [1 + e^2(j^2 + 4j + \tfrac{7}{2})] \quad (95)$$

$$B_2 = 2q_1(e/c)(j + 1) - 3(e^2/c^2)q_1 \times (j^2 + 4j + \tfrac{7}{2}) + 3(q_2/c^2) \quad (96)$$

$$B_3 = 6(e/c)q_2(j + 2) + 15(q_3/c^3) \quad (97)$$

$$B_4 = (15q_3/c^3)[2e(j + 3) + (e^2/c) \times (j^2 + 12j + \tfrac{43}{2})] + (105q_4/c^4) \quad (98)$$

$$B_5 = 210q_4(e/c^4)(j + 4) \quad (99)$$

Table 1 Modified Bessel functions of the first kind (from Ref. 8)

X	$I_0(X)$	$I_1(X)$	$I_2(X)$	$I_3(X)$	$I_4(X)$	$I_5(X)$	$I_6(X)$	$I_7(X)$	$I_8(X)$
0	1.0	0	0	0	0	0	0		
0.1	1.002502	0.0500625	0.00125104	$0.0^{4a}208463$	$0.0^{6}260547$	$0.0^{8}260525$	$0.0^{10}2170914$	$0.0^{12}155058$	$0.0^{15}96908$
0.2	1.010025	0.1005008	$0.0^{2}501669$	$0.0^{3}167084$	$0.0^{5}417501$	$0.0^{7}834723$	$0.0^{8}1390874$	$0.0^{10}198661$	$0.0^{12}248292$
0.3	1.022627	0.15169384	0.0113346	$0.0^{3}565671$	$0.0^{4}211888$	$0.0^{6}635189$	$0.0^{7}158712$	$0.0^{9}339961$	$0.0^{11}637228$
0.4	1.040402	0.204027	0.0202680	0.00134672	$0.0^{4}6720178$	$0.0^{5}2684495$	$0.0^{7}893981$	$0.0^{8}255241$	$0.0^{10}637748$
0.5	1.063483	0.257894	0.0319061	0.00264511	$0.0^{3}1648055$	$0.0^{5}822317$	$0.0^{6}3421236$	$0.0^{7}122051$	$0.0^{9}381078$
1.0	1.26607	0.565159	0.135748	0.0221684	0.00273712	0.0002714631	$0.0^{4}2248866$	$0.0^{5}159922$	$0.0^{7}9960624$
1.5	1.64672	0.981666	0.337835	0.0807741	0.0147382	0.00217056	0.000267769	$0.0^{4}284064$	$0.0^{5}264258$
2.0	2.279585	1.590637	0.688948	0.2127400	0.0507286	0.00982568	0.001600173	$0.0^{3}224639$	$0.0^{4}276994$

[a] 0.0^4 means 0.0000.

Case II: $e = 0$

When $e = 0$, then $c = 0$, and it can be shown that by setting $e = 0$ before integrating

$$\Delta a/\text{rev} = \Delta h/\text{rev} = -4\pi B r^2 (1 - d)^2 \rho_\pi \epsilon^{-q/2} I_0(q/2) \quad (100)$$

$$\Delta \tau/\text{rev} = -6\pi \tau B r (1 - d)^2 \rho_\pi \epsilon^{-q/2} I_0(q/2) \quad (101)$$

where ρ_π now is measured at the equatorial crossing.

It is interesting to note that Eq. (100) is in agreement with the results obtained by Fosdick[12] for drag decay by an oblate atmosphere, except for the $(1 - d)^2$ term. Since, in this work, the rotation of the atmosphere as well as its oblateness were taken into account, the $(1 - d)^2$ term is attributed to the rotation of the atmosphere. Since $d \approx \frac{1}{15} \cos i$, the rotation of the atmosphere contributes significantly to the drag decay.

Because of the $\cos i$, which appears in the expression for d, the atmospheric rotation decreases the decay rate (Δh/rev) for $0 \leq i < 90°$ (direct orbit) and increases the decay rate for $90° < i \leq 180°$ (retrograde orbit), as would be expected.

Summary

Accurate expressions are derived for the secular change in eccentricity, the semimajor axis, and the orbital period over one satellite revolution, because of the drag by an oblate, rotating atmosphere. Two cases were considered: $0 < e \leq 0.01$ and $e = 0$.

At $e = 0$, the expression for Δa/rev gives the time rate of decay of orbital radius per revolution (Δr/rev), that agrees with the Δr/rev obtained by Fosdick[12] except for a factor of $(1 - d)^2$, resulting from the rotation of the atmosphere, which Fosdick neglected. In fact, the expression for Δr/rev obtained in this work is the same as that for a circular, nonrotating atmosphere, except for a factor due to the oblateness and another factor due to the rotation of the atmosphere. Both of these are significant. Hence it can be concluded that both the oblateness and the rotation of the atmosphere are significant and cannot be neglected in an accurate theory of satellite motion.

The atmosphere was assumed to have the same flattening and angular velocity as the earth. Also, it was assumed to vary exponentially in the altitude range between the minimum and maximum altitudes experienced by the satellite in one revolution over an oblate earth. The exponential form of the atmosphere was expanded in a series in which terms through q^4 were retained. The q has values of about 0.9, 0.5, and 0.2 at altitudes of about 86 naut miles, 100 naut miles, and 200 naut miles, respectively, so that this series expansion is accurate within about 1.2%, 0.055%, and 0.0032%, again, respectively.

Other assumptions follow. Thus, it was assumed that the atmospheric lapse rate K is constant over the altitude range experienced by the satellite during one revolution. This is not quite true since the maximum altitude range experienced by a satellite in one revolution will be only about 80 naut miles at $e = 0.01$, the largest e considered in this work. Over this relatively small altitude range, however, it is quite feasible to choose K, which gives the best fit of the analytical expression for the atmospheric density to the true density curve. Another assumption is implicit in the dropping of some terms in the expression for the satellite air velocity, such that the expression is accurate to only about one part in 500 or about 0.2%.

Hence, it can be concluded that the resulting expressions for Δe/rev, Δa/rev, and $\Delta \tau$/rev can be expected to be very accurate. The predicted values of eccentricity, the semimajor axis, and orbital period after one revolution are even more accurate, because the error in the predicted value in "a," for example (neglecting other uncertainties, such as atmospheric variations from the nominal), is given by the following: relative error in predicted a is equal to [relative error in (Δa/rev)][(Δa/rev)/($a + \Delta a$/rev)]. This latter error is very small, due to the small factor (Δa/rev)/($a + \Delta a$/rev). It is probably negligible since it will be overshadowed by other uncertainties such as dispersions in the atmospheric density. This holds, similarly, for the predicted values of e and τ.

It is interesting to note that the atmospheric drag effects on the orbital period and the semimajor axis are less for direct orbits ($0° \leq i < 90°$) than for retrograde orbits ($90° < i \leq 180°$) because of the atmosphere's rotation.

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